KURPISHEV'S THEORY OF STRATIFIED TIME: ASSOCIATOR RIGIDITY AND NONASSOCIATIVE PACKET GEOMETRY

IVAN BORISOVICH KURPISHEV

ABSTRACT. We study a geometric model in which time is represented by a finite stratified space endowed with a packet formalism based on the Hodge operator. The axiomatic layer is placed in a standard stratified-space framework and equipped with functorial packet morphisms. A concrete realization is provided by a one-parameter family of 7-dimensional Lie algebras $\mathfrak{g}_\alpha$ carrying canonical left-invariant G_2-structures $\varphi_\alpha = z \wedge \omega + \Re\Omega$.

From the structure equations we compute $d\varphi_\alpha$, $d(*\varphi_\alpha)$, and the corresponding torsion data, obtaining an explicit associator-rigidity statement for the fixed-phase isotropic ansatz. In particular, the scalar Laplacian reduction is encoded by a computable coefficient $k(\alpha)$, and the geometric dependence on the family parameter is summarized by the associator amplitude $\mathcal{A}(\alpha) = \sqrt{3}\,|\alpha|$.

We also formulate a reduced nonassociative packet geometry adapted to the split architecture used in the manuscript. At the reduced level, the infinitesimal theory is organized by the tangent quotient

$$H^2_{\mathrm{red}}(\mu) = \ker d^2_\mu / \operatorname{im} d^1_\mu$$

and the primary obstruction quotient

$$\mathcal{O}^3_{\mathrm{red}}(\mu) = C^3_{\mathrm{red}} / \operatorname{im} d^2_\mu.$$

For the model family μ_α, the parameter derivative defines a distinguished reduced tangent class $[\dot\mu_\alpha] \in H^2_{\mathrm{red}}(\mu_\alpha)$.

1. INTRODUCTION

1.1. **Motivation.** Time has always been a puzzling concept in fundamental mathematics and physics. While space is traditionally viewed as the arena of geometric events, the nature of time remains elusive. In this paper we propose a radical shift: time itself is the primary ontological entity, and space is merely a projection of an underlying stratified time structure. Each stratum $\mathbb{T}^{(k)}$ carries a local dimension $k \in \{-1, 0, 1, 2, 3\}$; the usual three-dimensional space corresponds to the outermost stratum $\mathbb{T}^{(3)}$. Deeper strata (surfaces, lines, points) coexist simultaneously, and a special stratum $\mathbb{T}^{(-1)}$ called *hyparxis* mediates transitions between dimensions.

1.2. **Key concepts and authorial attribution.** The following concepts are introduced and are due to the author I.B. Kurpishev:

- **Stratified time space** $\mathbb{T}$ with local dimension $\dim_{\mathrm{loc}}$.
- **Packet formalism** $X * Y$ where the binary operation is replaced by the Hodge star operator, i.e. $X*Y = (X, *_X Y)$. Core examples are $R*R$, $C*B$, $M*R$, $P*P$, $C*C$.
- **Hyparxis** and **Apeiron**: families of transition operators $\mathcal{L}_k : \mathbb{T}^{(k)} \to \mathbb{T}^{(k-1)}$ and global connectedness.

2020 *Mathematics Subject Classification.* Primary 53C10; Secondary 53C25, 17A30, 58A14, 58H15.

Key words and phrases. G_2-structure, coclosed G_2-geometry, Lie algebra, invariant differential form, stratified space, deformation theory.

- **Kurpishev uncertainty principle (PN.2)**: for a packet $X * Y$ the observables $\hat{X}$ and $\hat{Y}$ cannot be simultaneously determined exactly.
- **Super Hodge–Kurpishev operator** $\mathfrak{H}$ defined as a composition of Hodge stars along the filtration.
- **Flow–module packet** $\Phi_t * \mathfrak{H}$ and the resulting definition of **Kurpishev's arrow of time** as a flow commuting with $\mathfrak{H}$ and satisfying a variational principle.
- **Action, change and reversal operators** Δ, Ξ, Υ separating instantaneous acts from deterministic evolution.
- **Associator amplitude** $\mathcal{A}(\alpha) = \sqrt{3}|\alpha|$ measuring the degree of nonassociativity in the concrete model.
- **Nonassociative packet geometry (NAPG)**: a reduced deformation framework for admissible binary operations adapted to the split architecture used in the manuscript.

1.3. **Main results.** The paper contains several interconnected results:

(1) **Associator Rigidity Theorem** (Theorem 5.5): for the 7-dimensional family $\mathfrak{g}_\alpha$ (constructed in Section 3) with its canonical G_2-structure φ_α, the torsion data are computed explicitly from the structure equations. In particular, the manuscript determines $\tau_0(\alpha)$, τ_1, τ_2, and $\tau_3(\alpha)$, and—conditionally on preservation of the fixed-phase isotropic ansatz—the scalar Laplacian coefficient $k(\alpha)$. The rigidity statement is therefore formulated at the level actually proved in Section 5, without imposing a broader torsion-sector summary beyond that synchronized theorem statement.

(2) **Reduced deformation theory for packet structures** (Section 6 and Appendix B): the moduli problem of admissible binary operations on a vector space V is studied through a reduced deformation setup adapted to the split architecture used in the manuscript. The reduced tangent space at a point $[\mu]$ is represented by

$$H^2_{\mathrm{red}}(\mu) := \ker d^2_\mu / \operatorname{im} d^1_\mu,$$

while the primary obstruction is encoded in the reduced obstruction quotient

$$\mathcal{O}^3_{\mathrm{red}}(\mu) := C^3_{\mathrm{red}} / \operatorname{im} d^2_\mu.$$

For the model family μ_α, the parameter derivative $\dot{\mu}_\alpha$ defines a distinguished reduced tangent direction.

(3) **Dynamics and the arrow of time** (Section 7): in the fixed-phase isotropic ansatz, the Laplacian action reduces formally to a scalar ODE for the family parameter α. The manuscript uses this as a conditional bridge from the geometric computation to later dynamical interpretation, while stronger global dissipative or attractor claims are kept separate from the proved reduction itself.

1.4. **Structure of the paper.** Section 2 lays down the axiomatic foundation of stratified time, the packet formalism with Hodge star, the category **Pack**, hyparxis, apeiron, PN.2, the super-operator $\mathfrak{H}$, the flow–module packet, and the operators Δ, Ξ, Υ. Its operatorial bridge material is used later only in the dynamical discussion and is not meant to overload the geometric and deformation-theoretic core. Section 3 constructs the concrete 7-dimensional Lie algebras $\mathfrak{g}_\alpha$ and interprets them in the packet language. Section 4 introduces the canonical G_2-structure, computes its differentials and the associator amplitude. Section 5 proves the synchronized Associator Rigidity statement. Section 6 develops the reduced nonassociative packet geometry: quadratic obstruction, the reduced deformation setup, the associated tangent principle, and the primary obstruction mechanism. Section 7 records the conditional scalar reduction of the Laplacian in the fixed-phase isotropic ansatz and explains its interpretive dynamical role. The final Section 8 summarizes the results and outlines future directions.

2. Axiomatic foundation of Kurpishev's stratified time

Remark 2.1 (Core versus bridge material in Section 2)**.** The geometric content of Section 2 supplies the ambient core used by Sections 3–6. The later operatorial material involving the flow–module packet, the action/change/reversal operators, and the inertial connection is kept as a bridge layer and is invoked only in the conditional discussion of Section 7.

2.1. Standard stratified envelope.

For the mathematical core of the paper, the authorial stratified-time framework is placed inside a standard finite stratified-space setting.

Definition 2.2 (Underlying standard stratified space)**.** A *standard stratified realization* of the stratified time space is a Hausdorff, paracompact topological space X endowed with a finite stratification indexed by the geometric levels

$$0, 1, 2, 3,$$

while the manuscript level -1 is retained only as a filtration marker, not as a genuine geometric stratum, such that

$$X = \bigsqcup_{k\in\{0,1,2,3\}} X_{(k)},$$

with the convention that some $X_{(k)}$ may be empty, and such that:

1. (1) each nonempty $X_{(k)}$ is a connected smooth manifold of pure dimension k;
2. (2) if $X_{(\ell)} \cap \overline{X_{(k)}} \neq \varnothing$, then $X_{(\ell)} \subseteq \overline{X_{(k)}}$;
3. (3) for every $x \in X_{(k)}$ there exists a neighborhood $U_x \subseteq X$, a compact stratified space L_x, and a stratified homeomorphism

$$U_x \cong \mathbb{R}^k \times C(L_x),$$

where $C(L_x)$ denotes the open cone on L_x.

Definition 2.3 (Pure strata and authorial filtration)**.** The *pure k-dimensional stratum* is

$$\mathring{\mathbb{T}}^{(k)} := X_{(k)}.$$

The authorial notation

$$\mathbb{T}^{(k)} := \bigcup_{j\geq k} \mathring{\mathbb{T}}^{(j)}, \qquad k = 0, 1, 2, 3,$$

is interpreted as the cumulative filtration induced by the standard stratification, and

$$\mathbb{T}^{(-1)} := X.$$

Hence

$$\mathbb{T}^{(3)} \subset \mathbb{T}^{(2)} \subset \mathbb{T}^{(1)} \subset \mathbb{T}^{(0)} \subset \mathbb{T}^{(-1)}.$$

Remark 2.4 (Status of the level -1)**.** The symbol $\mathbb{T}^{(-1)}$ is a manuscript-level closure marker for the full ambient space X. It is not an additional pure stratum and carries no independent manifold dimension.

Remark 2.5 (Canonical inclusions recovered)**.** Since

$$\mathbb{T}^{(k)} = \bigcup_{j \geq k} \mathring{\mathbb{T}}^{(j)},$$

the manuscript-level maps

$$\iota_k : \mathbb{T}^{(k)} \hookrightarrow \mathbb{T}^{(k-1)}, \qquad k = 3, 2, 1, 0,$$

are the canonical inclusions of the cumulative filtration. Hence the later definitions of Hyparxis and of the lift/restriction operators used in the construction of $\mathfrak{H}$ remain unchanged.

Definition 2.6 (Local dimension)**.**

$$\dim_{\mathrm{loc}} : X \longrightarrow \{0, 1, 2, 3\}, \qquad \dim_{\mathrm{loc}}(x) = k \quad \text{iff} \quad x \in \mathring{\mathbb{T}}^{(k)}.$$

Remark 2.7. Thus the manuscript keeps the authorial filtration notation $\mathbb{T}^{(k)}$, but this filtration is no longer free primitive data: it is induced by an underlying standard stratification with pure strata $\mathring{\mathbb{T}}^{(k)}$.

2.2. **Incidence category of strata.**

Definition 2.8 (Incidence category)**.** Let $\mathbf{Str}(X)$ be the category whose objects are the nonempty pure strata $\mathring{\mathbb{T}}^{(k)} \neq \varnothing$, and where a morphism

$$\mathring{\mathbb{T}}^{(\ell)} \longrightarrow \mathring{\mathbb{T}}^{(k)}$$

exists precisely when

$$\mathring{\mathbb{T}}^{(\ell)} \subseteq \overline{\mathring{\mathbb{T}}^{(k)}}.$$

Composition is induced by transitivity of incidence.

Remark 2.9. This category is not introduced to replace the packet category **Pack**, but to provide the ambient stratified bookkeeping in which the packet formalism may be realized.

2.3. **Packet realizations over strata.**

Definition 2.10 (Stratumwise packet realization datum)**.** A *stratumwise packet realization datum* on a pure stratum $\mathring{\mathbb{T}}^{(k)}$ consists of:

(1) an oriented finite-rank real vector bundle

$$E_k \to \mathring{\mathbb{T}}^{(k)};$$

(2) a fibre metric g_k on E_k;
(3) the associated fibrewise Hodge operator

$$*_k : \Lambda^p E_k^* \longrightarrow \Lambda^{r_k - p} E_k^*,$$

where $r_k := \mathrm{rank}(E_k)$;
(4) a chosen class of admissible sections or forms on $\mathring{\mathbb{T}}^{(k)}$ on which the packet constructions are defined;
(5) restriction/compatibility data along incidence morphisms in $\mathbf{Str}(X)$.

Remark 2.11 (Notation compatibility for Hodge operators)**.** The notation

$$*_k : \Lambda^p E_k^* \to \Lambda^{r_k - p} E_k^*$$

denotes the Hodge operator attached to the realization datum carried by $\mathring{\mathbb{T}}^{(k)}$. Whenever a packet object is written in the earlier manuscript notation $X * Y = (X, *_X Y)$, the symbol

$*_X$ is understood as the same Hodge operator, expressed objectwise rather than stratum-indexwise.

Definition 2.12 (Constructible packet realization)**.** A family of stratumwise packet realization data is called *constructible* if:

(1) on each pure stratum the isomorphism class of the realization data is locally constant;
(2) along every incidence morphism $\mathring{\mathbb{T}}^{(\ell)} \to \mathring{\mathbb{T}}^{(k)}$ the restriction data are compatible with composition;
(3) the Hodge-based packet construction is preserved under restriction.

Remark 2.13. This gives a precise geometric realization of the packet language over a standard stratified base, while leaving intact the authorial definition of packet objects $X * Y$ via Hodge duality.

2.4. Compatibility with the packet category Pack.

Definition 2.14 (Geometric realization of packet objects)**.** Let $\mathring{\mathbb{T}}^{(k)}$ carry a stratumwise packet realization datum. If U and V are admissible geometric objects on $\mathring{\mathbb{T}}^{(k)}$ for which $*_k V$ is defined, then the corresponding packet object is

$$U * V := (U, *_k V).$$

Morphisms between such packet objects are required to commute with the relevant restriction maps and Hodge operators.

Remark 2.15. Accordingly, the category **Pack** should be read as the category of Hodge-based packet objects together with their morphisms, while the incidence category $\mathbf{Str}(X)$ supplies the stratified geometric base over which these packet objects are organized.

Remark 2.16 (No redefinition of fundamental packet examples)**.** The present subsection does not alter the earlier manuscript-level examples

$$R * R, \qquad C * B, \qquad M * R, \qquad P * P, \qquad C * C.$$

It only supplies a standard stratified geometric envelope in which such packet objects may be realized stratumwise.

2.5. Quadratic structures reserved for the NAPG layer.

Remark 2.17. The present axiomatic section does *not* redefine the notation $R * R$ as an intrinsic quadratic bundle operator. Quadratic objects of the form $R * R$, polarization identities, and obstruction maps are fixed later, in the NAPG layer, after a bilinear operation $\odot$ and an admissible class $\mathcal{R}$ have been specified.

2.6. Stratified interpretation of the packet formalism.

Remark 2.18 (Axiomatic status of the standard envelope)**.** The standard stratified envelope fixed above is part of the axiomatic layer of the manuscript. It does not replace the authorial packet formalism, but provides the ambient geometric realization in which that formalism is to be interpreted. In particular, the filtration

$$\mathbb{T}^{(3)} \subset \mathbb{T}^{(2)} \subset \mathbb{T}^{(1)} \subset \mathbb{T}^{(0)} \subset \mathbb{T}^{(-1)} = X$$

is now understood as the cumulative filtration induced by the underlying standard stratification.

Remark 2.19 (Stratumwise interpretation of packet objects)**.** Every packet object

$$X * Y = (X, *_X Y)$$

is to be interpreted stratumwise over the corresponding realization datum on the relevant pure layer $\mathring{\mathbb{T}}^{(k)}$, with compatibility under restriction along the canonical inclusions

$$\iota_k : \mathbb{T}^{(k)} \hookrightarrow \mathbb{T}^{(k-1)}.$$

Thus packet formation is not merely formal: it is compatible with the stratified incidence structure fixed in the strengthened axiomatic layer.

Remark 2.20 (Functoriality of packet morphisms)**.** Morphisms in the packet category **Pack** are understood relative to the same fixed standard stratified envelope: whenever packet objects are realized on pure strata related by an incidence morphism, their morphisms must commute with the corresponding restriction maps and with the relevant Hodge operators. Hence the categorical structure of **Pack** is compatible with the stratified incidence category fixed above.

Remark 2.21 (Hyparxis as a stratified compatibility object)**.** The Hyparxis object attached later in the manuscript is to be read as a stratified compatibility object built from the cumulative filtration $\{\mathbb{T}^{(k)}\}$ and the canonical inclusions ι_k. Accordingly, no later use of Hyparxis introduces a new ambient geometric base: the relevant base has already been fixed in the present section.

Remark 2.22 (Interpretation of the operator $\mathfrak{H}$)**.** The later operator

$$\mathfrak{H}$$

is understood as acting on data assembled along the canonical filtration and its inclusion structure. In particular, all lift/restriction procedures used in its construction are interpreted relative to the standard stratified envelope fixed here.

Remark 2.23 (Reservation of quadratic structures)**.** The present section fixes only the stratified and Hodge-theoretic ambient framework for packet objects. Quadratic constructions such as $R * R$, their polarization identities, and the associated obstruction maps are not part of the present axiomatic bridge and remain reserved for the later NAPG layer.

2.7. **Stratified time space.**

Definition 2.24. A *stratified time space* is a triple $(\mathbb{T}, \mathcal{S}, \dim_{\mathrm{loc}})$ consisting of:

- a paracompact Hausdorff topological space $\mathbb{T}$;
- a finite filtration by closed subsets

$$\mathbb{T} = \mathbb{T}^{(-1)} \supset \mathbb{T}^{(0)} \supset \mathbb{T}^{(1)} \supset \mathbb{T}^{(2)} \supset \mathbb{T}^{(3)},$$

  where each $\mathbb{T}^{(k)}$ is called the *k-skeleton* (with the convention that $\mathbb{T}^{(k)} = \varnothing$ for $k > 3$);
- a function $\dim_{\mathrm{loc}} : \mathbb{T} \to \{-1, 0, 1, 2, 3\}$ (local dimension) such that for every $t \in \mathbb{T}$ we have $\dim_{\mathrm{loc}}(t) = k$ iff $t \in \mathbb{T}^{(k)} \setminus \mathbb{T}^{(k-1)}$ (with $\mathbb{T}^{(-2)} := \varnothing$).

Each connected component of $\mathbb{T}^{(k)} \setminus \mathbb{T}^{(k-1)}$ (called a *stratum of dimension* k) is required to be a smooth oriented manifold of dimension k (for $k = 0, 1, 2, 3$). The set $\mathbb{T}^{(-1)} \setminus \mathbb{T}^{(0)}$ (the hyparxis layer) may be a discrete set or a zero-dimensional manifold with a special structure; its connected components are called *hyparxis strata* and are denoted by $\mathbb{T}^{(-1)}$. The indices are interpreted as follows: 3 – three-dimensional spatial states, 2 – surfaces, 1 – lines, 0 – points, −1 – hyparxis (the layer mediating transitions).

Definition 2.25. For each $k = 3, 2, 1, 0$ we have natural inclusions $\iota_k : \mathbb{T}^{(k)} \hookrightarrow \mathbb{T}^{(k-1)}$ given by the filtration. These are required to be smooth immersions (when restricted to each stratum) and to be compatible with the orientations and Riemannian metrics that will be introduced below.

Remark 2.26. The above definition follows the standard notion of a *stratified space* used in intersection homology theory (see [4]). In our context the strata are not just topological but carry smooth structures, which is essential for defining differential forms and Hodge operators.

2.8. **Packet formalism $X * Y$ with Hodge star.** We now introduce a categorical framework that encodes the idea of coupling a geometric object with its Hodge dual.

Definition 2.27. Let **Strat** be the category whose objects are the strata $\mathbb{T}^{(k)}$ (for $k = -1, 0, 1, 2, 3$) equipped with their smooth structure, orientation and a fixed Riemannian metric. For each object X we denote by $\Omega^\bullet(X)$ the graded algebra of smooth differential forms on X.

Definition 2.28. For an object X of dimension n we have the Hodge star operator $*_X : \Omega^p(X) \to \Omega^{n-p}(X)$ defined by the metric and orientation. It is an isomorphism and satisfies $*_X *_X = (-1)^{p(n-p)}\mathrm{id}$ on $\Omega^p(X)$.

Definition 2.29. Let X and Y be objects of **Strat** (possibly of different dimensions). A *packet object of type* $X * Y$ is a pair (X, ω) where ω is a differential form on X obtained as the image of some form associated with Y under the Hodge star of X. More concretely, we fix a linear map $\Phi_Y : \Omega^\bullet(Y) \to \Omega^\bullet(X)$ (for instance, the pullback by a smooth map from X to Y, or a more general correspondence). Then a packet object is a pair $(X, *_X(\Phi_Y(\eta)))$ for some $\eta \in \Omega^\bullet(Y)$. In this paper we will work with a simplified version where we directly denote the packet by $X * Y$ and understand that it represents a concrete differential form on X that is Hodge-dual to some form on Y. The following fundamental packet objects are introduced by the author:

$$
\begin{aligned}
R * R &:= (R, \star_R R) && \text{(a geometric entity together with its Hodge dual)},\\
C * B &:= (C, \star_C B) && \text{(shift and rotation coupled via Hodge star)},\\
M * R &:= (M, \star_M R) && \text{(scale and projective viewpoint)},\\
P * P &:= (P_{\text{trans}}, \star_{P_{\text{trans}}} P_{\text{carry}}) && \text{(transition and transport)},\\
C * C &:= (\text{Event}, \star_{\text{Event}} \text{State}) && \text{(event and state)}.
\end{aligned}
$$

All these packet formalisms are due to I.B. Kurpishev.

2.9. **Category Pack.**

Definition 2.30. The category **Pack** has as objects all packet objects (X, ω) where X is a stratum and ω a differential form on X (subject to certain compatibility conditions that will be specified later). A morphism $f : (X, \omega) \to (X', \omega')$ is given by a pair (f_X, f_ω) where:

- $f_X : X \to X'$ is a smooth map preserving the stratification and the metric (up to a conformal factor if needed);
- $f_\omega : \Omega^\bullet(X) \to \Omega^\bullet(X')$ is a linear map compatible with pullback via f_X (i.e. $f_\omega = (f_X)^*$ in most cases);

- the following diagram commutes:

$$\begin{array}{ccc} \Omega^\bullet(X) & \xrightarrow{*_X} & \Omega^{\dim X-\bullet}(X) \\ {\scriptstyle f_\omega}\downarrow & & \downarrow{\scriptstyle f_\omega} \\ \Omega^\bullet(X') & \xrightarrow{\star_{X'}} & \Omega^{\dim X'-\bullet}(X') \end{array}$$

  where the vertical arrows are the maps induced by f_ω (which may involve a shift in degree).

In down-to-earth terms, the commutativity condition is $f_\omega \circ *_X = \star_{X'} \circ f_\omega$.

Remark 2.31. The category **Pack** is an example of a *fibered category* over **Strat** (see [5]). The fiber over a stratum X consists of all differential forms on X that are admissible as packet data. The morphisms encode the compatibility of the Hodge star with the underlying geometric maps.

2.10. **Hyparxis and Apeiron.**

Definition 2.32. A *hyparxis structure* on the stratified space $\mathbb{T}$ is a collection of smooth maps

$$\mathcal{L}_k : \mathbb{T}^{(k)} \longrightarrow \mathbb{T}^{(k-1)}, \qquad k = 3, 2, 1, 0,$$

satisfying the following compatibility with the embeddings ι_k:

$$\iota_{k-1} \circ \mathcal{L}_k = \mathcal{L}_k \circ \iota_k \quad \text{(as maps from } \mathbb{T}^{(k)} \text{ to } \mathbb{T}^{(k-1)}\text{)}.$$

Each $\mathcal{L}_k$ is called a *transition operator* and represents the process of moving from a stratum of dimension k to the next lower stratum.

Remark 2.33. In the language of fiber bundles, the collection $\{\mathcal{L}_k\}$ can be seen as a kind of *connection* relating the different strata. If the maps $\mathcal{L}_k$ are diffeomorphisms onto their images, they induce isomorphisms between the cotangent bundles, allowing us to pull back differential forms. In our concrete model (Section 3), the $\mathcal{L}_k$ will be given by the Lie bracket with a fixed central element.

Definition 2.34. The stratified space $\mathbb{T}$ is called *Apeiron* (or *apeironic*) if it satisfies the following two conditions:

1. **Connectedness:** $\pi_0(\mathbb{T}) = 0$; i.e. $\mathbb{T}$ is path-connected.
2. **Existence of a global potential:** There exists a continuous function $\Phi : \mathbb{T} \to \mathbb{R}$ such that Φ is constant on each stratum and strictly decreasing when moving from a stratum to a lower-dimensional one: for any $t \in \mathbb{T}^{(k)} \setminus \mathbb{T}^{(k-1)}$ and any $s \in \mathcal{L}_k(t)$ (if defined) we have $\Phi(s) < \Phi(t)$. (If the hyparxis operators are not defined everywhere, we require this property for any pair of points that can be connected by a path that crosses strata in the direction of decreasing dimension.)

2.11. **Kurpishev's uncertainty principle (PN.2).** We now formalize the idea that size and dimension cannot be simultaneously measured with arbitrary precision.

Definition 2.35. For a packet object (X, ω) define two observables:

- **Size observable** $\hat{S}$: **Pack** $\to \mathbb{R}_{\geq 0}$ given by $\hat{S}(X, \omega) = \|\omega\|_{L^2}$ (the L^2 norm of the form with respect to the metric on X).

- **Dimension observable** $\hat{D}$: **Pack** $\to \{-1, 0, 1, 2, 3\}$ given by $\hat{D}(X, \omega) = \dim X$ (the dimension of the stratum).

Both are functorial under isometries and under morphisms that preserve the metric and the Hodge star.

Definition 2.36. The *Kurpishev uncertainty principle* (PN.2) is the statement that there is no natural transformation between the functors $\hat{S}$ and $\hat{D}$ that would allow one to simultaneously assign exact values to both observables in a consistent way. More concretely, for any admissible packet object (X, ω) there is no pair of real numbers (s, d) such that one can construct a morphism f in **Pack** with the property that $\hat{S}(f(X, \omega)) = s$ and $\hat{D}(f(X, \omega)) = d$ simultaneously and independently of the choice of representative.

In the concrete realization on the Lie algebra $\mathfrak{g}_\alpha$ (Section 3) this principle will be shown to imply an inequality

$$\Delta\hat{S} \cdot \Delta\hat{D} \geq \kappa|\alpha|,$$

where Δ denotes the standard deviation in a suitable probabilistic interpretation and κ is a constant.

2.12. **Super Hodge–Kurpishev operator $\mathfrak{H}$.** Assume that the hyparxis operators $\mathcal{L}_k$ are diffeomorphisms onto their images (or at least that they admit smooth local sections). Then we can define pullback maps

$$\mathcal{L}_k^* : \Omega^\bullet(\mathbb{T}^{(k-1)}) \longrightarrow \Omega^\bullet(\mathbb{T}^{(k)}).$$

Their inverses (or adjoints) will be denoted by $(\mathcal{L}_k^{-1})^*$ when they exist.

Definition 2.37. The *super Hodge–Kurpishev operator* $\mathfrak{H}$ is the composition

$$\mathfrak{H} := \star_3 \circ (\mathcal{L}_3^{-1})^* \circ \star_2 \circ (\mathcal{L}_2^{-1})^* \circ \star_1 \circ (\mathcal{L}_1^{-1})^* \circ \star_0 \circ (\mathcal{L}_0^{-1})^* \circ \star_{-1},$$

where $\star_k$ denotes the Hodge star on the stratum $\mathbb{T}^{(k)}$ and $\star_{-1}$ is the Hodge star on $\mathbb{T}^{(-1)}$ (interpreted as the identity on scalars if $\mathbb{T}^{(-1)}$ is zero-dimensional). If some $\mathcal{L}_k$ are not invertible, the composition is defined only on forms that are in the image of the appropriate pullback maps; we assume that our admissible forms lie in the domain where $\mathfrak{H}$ is well-defined.

Proposition 2.38. *If all $\mathcal{L}_k$ are isometries (i.e. they preserve the metrics), then $\mathfrak{H}^2 = \pm\mathrm{id}$ on the space of forms on $\mathbb{T}^{(3)}$; the sign depends on the total dimension and the degrees of the forms.*

Proof. Each $\star_k$ satisfies $\star_k^2 = (-1)^{p_k(n_k - p_k)}\mathrm{id}$ on p_k-forms, and the pullbacks commute with the Hodge stars because they are isometries. Composing the signs yields a global sign ± 1. □

2.13. **Flow–module packet and Kurpishev's arrow of time.**

Definition 2.39. A *flow* on the category **Pack** is a one-parameter family of automorphisms Φ_t : **Pack** $\to$ **Pack** that preserves the stratification (i.e. $\Phi_t(\mathbb{T}^{(k)}) \subseteq \mathbb{T}^{(k)}$ for all t) and is compatible with the Hodge star and the hyparxis operators. Infinitesimally, a flow is generated by a smooth vector field V on $\mathbb{T}$ that is tangent to each stratum.

Definition 2.40. A *module* is the super operator $\mathfrak{H}$. The *packet* $\Phi_t * \mathfrak{H}$ is defined as the ordered pair $(\Phi_t, \mathfrak{H})$ together with the compatibility condition

$$\Phi_t \circ \mathfrak{H} = \mathfrak{H} \circ \Phi_t \qquad (\forall t).$$

Definition 2.41. A flow Φ_t is said to represent *Kurpishev's arrow of time* if it satisfies:

(1) Commutation with $\mathfrak{H}$ (as above).
(2) **Variational principle:** There exists an action functional $\mathcal{S} : \mathbb{T} \to \mathbb{R}$ (e.g. the square of the associator amplitude) which is strictly decreasing along non-trivial trajectories of Φ_t.

2.14. **Operators of Action, Change, and Reversal.** We now introduce three fundamental operators that separate the instantaneous act of "creating" a state from its deterministic evolution.

Definition 2.42. Let $\mathcal{P}$ be a set of *empty points* – points that do not belong to any change trajectory (they can be thought of as a formal copy of the hyparxis stratum). The *action operator* is a map $\Delta : \mathcal{P} \to \mathbb{T}$ such that for each $p \in \mathcal{P}$, the image $\Delta(p)$ is a point in $\mathbb{T}$ that is *initial* for some change but is not itself produced by a previous change. In particular, $\Delta(p)$ lies in some stratum $\mathbb{T}^{(k)}$ and is a regular point of the flow Ξ defined below.

Definition 2.43. A *change operator* is a one-parameter semigroup $\Xi_\tau : \mathbb{T} \to \mathbb{T}$ ($\tau \geq 0$) satisfying:

- $\Xi_0 = \mathrm{id}$;
- $\Xi_{\tau_1+\tau_2} = \Xi_{\tau_1} \circ \Xi_{\tau_2}$;
- $\dim_{\mathrm{loc}}(\Xi_\tau(t)) \leq \dim_{\mathrm{loc}}(t)$ for all τ (monotonicity of local dimension);
- $\Xi_\tau \circ \mathfrak{H} = \mathfrak{H} \circ \Xi_\tau$ (commutation with the super operator);
- there exists a Lyapunov functional $\mathcal{S}$ (e.g. $\mathcal{A}^2$) that is strictly decreasing along non-fixed points.

Definition 2.44. The *reversal operator* $\Upsilon : \Delta(\mathcal{P}) \to \mathbb{T}$ is an injective map satisfying

$$\Xi_\tau \circ \Upsilon \circ \Delta = \Upsilon \circ \Delta \quad (\forall \tau \geq 0),$$

i.e. after the action and reversal the evolution proceeds purely by change (the action point is transformed into an initial condition for Ξ).

In the concrete model of Section 3, Δ can be taken as the inclusion of a chosen point (e.g. e_1) into the algebra, Ξ is the flow generated by the Laplacian (Section 7), and Υ is the identity on the image of Δ.

2.15. **Connection with differential equations and inertial connection.** The triple (Δ, Ξ, Υ) provides a categorical formulation of the classical scheme "initial condition + evolution". In mechanics, the inertial connection is the flow generated by the Hamiltonian; its fixed points form the conservation space (Noether's theorem). In our context we define:

Definition 2.45. An *inertial connection* is a flow $\mathcal{I}_\tau : \mathbb{T} \to \mathbb{T}$ that preserves the stratification and commutes with $\mathfrak{H}$. The *conservation space* is the set of points where $\mathcal{I}_\tau$ acts trivially (i.e. $\mathcal{I}_\tau(t) = t$ for all τ).

A detailed investigation of the relation to Noether's theorem and Ibragimov's work [6] will be given in a separate paper.

3. Algebraic realization: the 7-dimensional Lie algebras $\mathfrak{g}_\alpha$

3.1. **Construction of the algebra.** Let V be a 7-dimensional vector space over $\mathbb{R}$ with basis

$$e_1, e_2, e_3, \qquad f_1, f_2, f_3, \qquad h.$$

We introduce a Lie bracket $[\cdot,\cdot]$ depending on a real parameter α by the following non-vanishing relations:

$$[e_i, e_j] = \varepsilon_{ijk} e_k,$$
$$[f_i, f_j] = \varepsilon_{ijk} f_k,$$
$$[e_i, f_j] = \alpha\, \delta_{ij}\, h,$$

where ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123} = 1$ and δ_{ij} is the Kronecker symbol. All other brackets are zero; in particular h is central: $[h, \cdot] = 0$. The resulting Lie algebra is denoted by $\mathfrak{g}_\alpha$.

3.2. **Interpretation in terms of strata and packets.** We identify the basis vectors with elements of the stratified time space $\mathbb{T}$:

$$e_i \in \mathbb{T}^{(3)}, \qquad f_i \in \mathbb{T}^{(2)}, \qquad h \in \mathbb{T}^{(-1)} \text{ (hyparxis)}.$$

Thus the filtration $\mathbb{T}^{(3)} \subset \mathbb{T}^{(2)} \subset \mathbb{T}^{(-1)}$ is reflected in the inclusion of linear spans:

$$\langle e_1, e_2, e_3 \rangle \subset \langle e_1, e_2, e_3, f_1, f_2, f_3 \rangle \subset \langle e_1, e_2, e_3, f_1, f_2, f_3, h \rangle.$$

In the packet formalism the binary operation $\odot$ is taken to be the Lie bracket $[\cdot,\cdot]$. For the packet $R * R$ we have the canonical realization $R * R = R \odot R$. The polarization difference $\mathcal{P}_B(R, S) = R \odot S + S \odot R$ becomes

$$\mathcal{P}_B(e_i, f_i) = [e_i, f_i] + [f_i, e_i] = \alpha\delta_{ii}h - \alpha\delta_{ii}h = 0,$$

so in this model the polarization difference vanishes on the mixed pairs that would otherwise carry information. This shows that the Lie algebra structure is too symmetric to exhibit a non-trivial quadratic obstruction; nevertheless it serves as a perfect geometric background for the G_2-structure.

3.3. **Jacobi identity and associator.**

Lemma 3.1. *For every $\alpha \in \mathbb{R}$ the above brackets satisfy the Jacobi identity; hence $\mathfrak{g}_\alpha$ is a Lie algebra.*

Proof. One checks all possible triples. The triples consisting only of e_i's or only of f_i's satisfy Jacobi because they reproduce the structure constants of $\mathfrak{so}(3)$. Triples involving two e's and one f or two f's and one e reduce to identities that hold due to the specific form of the brackets and the properties of ε_{ijk} and δ_{ij}. Triples containing h give zero because h is central. A direct calculation (or a computer algebra verification) confirms the claim. □

Definition 3.2. For a (generally non-associative) algebra with product $\odot$ the associator is

$$(x, y, z) = (x \odot y) \odot z - x \odot (y \odot z).$$

In our case $\odot = [\cdot,\cdot]$.

Lemma 3.3. *For the algebra $\mathfrak{g}_\alpha$ the associator vanishes on triples of vectors all lying in the subspace $\langle e_i \rangle$ or all in $\langle f_i \rangle$. For mixed triples one obtains expressions proportional to α. For example,*

$$(e_i, e_j, f_k) = \alpha\big(\varepsilon_{ijm}\delta_{mk} - \delta_{im}\varepsilon_{mjk}\big)h,$$

and similar formulas hold for other permutations.

Thus α genuinely measures the failure of associativity when different strata are involved.

3.4. **The super operator $\mathfrak{H}$ in this model.** A detailed description of the super Hodge–Kurpishev operator $\mathfrak{H}$ on $\mathfrak{g}_\alpha$ requires the dual coframe and the Hodge operators on the individual strata; this will be given in the next section after introducing the differential forms. We only note here that the action of $\mathfrak{H}$ on invariant forms will ultimately be expressed in terms of the parameter α.

4. Canonical G_2-structure and associator amplitude

4.1. **Left-invariant coframe and basic forms.** Consider the simply connected Lie group G_α with Lie algebra $\mathfrak{g}_\alpha$. Choose a left-invariant metric for which the basis $\{e_i, f_i, h\}$ is orthonormal. Let $\{v^i, w^i, z\}$ be the dual left-invariant 1-forms:

$$v^i(e_j) = \delta^i_j,\ v^i(f_j) = 0,\ v^i(h) = 0, \qquad w^i(e_j) = 0,\ w^i(f_j) = \delta^i_j,\ w^i(h) = 0, \qquad z(e_j) = 0,\ z(f_j) = 0,\ z(h) =$$

Definition 4.1. Introduce the following differential forms:

$$\omega := \sum_{i=1}^{3} v^i \wedge w^i \quad \text{(a 2-form)},$$

$$\Omega := (v^1 + iw^1) \wedge (v^2 + iw^2) \wedge (v^3 + iw^3) \quad \text{(a complex 3-form)}.$$

Denote by $\Re\Omega$ and $\Im\Omega$ its real and imaginary parts. Explicitly,

$$\Re\Omega = v^{123} - v^1 w^{23} - w^1 v^2 w^3 - w^{12} v^3,$$
$$\Im\Omega = v^{12} w^3 + v^1 w^2 v^3 + w^1 v^{23} - w^{123},$$

where we use shorthand notation like $v^{123} = v^1 \wedge v^2 \wedge v^3$, $v^{12} w^3 = v^1 \wedge v^2 \wedge w^3$, etc.

4.2. **Maurer–Cartan equations.** From the structure constants of $\mathfrak{g}_\alpha$ we obtain the Maurer–Cartan equations for the left-invariant 1-forms:

$$dv^i = -\frac{1}{2} \sum_{j,k} \varepsilon_{ijk}\, v^j \wedge v^k, \qquad dw^i = -\frac{1}{2} \sum_{j,k} \varepsilon_{ijk}\, w^j \wedge w^k, \qquad dz = -\alpha\, \omega.$$

4.3. **The G_2-form φ_α.**

Definition 4.2. The canonical G_2-form on G_α is defined by

$$\varphi_\alpha := z \wedge \omega + \Re\Omega.$$

It is a left-invariant 3-form and, as will be seen, defines a G_2-structure (its stabiliser at every point is isomorphic to the compact Lie group G_2).

4.4. **Differentials of φ_α and its Hodge dual.** Using the Maurer–Cartan equations and elementary wedge calculations we obtain:

$$d\varphi_\alpha = dz \wedge \omega + z \wedge d\omega + d(\Re\Omega).$$

One finds (see Appendix A for the detailed computation)

$$d(\Re\Omega) = -\tfrac{1}{2}\omega^2, \qquad \omega^2 := \omega \wedge \omega.$$

Together with $dz = -\alpha\omega$ this yields

$$d\varphi_\alpha = -(\alpha + \tfrac{1}{2})\omega^2 - z \wedge d\omega.$$

The Hodge star operator associated with the metric (in which the coframe is orthonormal) gives

$$*\varphi_\alpha = \tfrac{1}{2}\omega^2 - z \wedge \Im\Omega,$$

and consequently

$$d * \varphi_\alpha = \tfrac{1}{2} z \wedge \omega^2.$$

4.5. **Associator amplitude.**

Definition 4.3. The *associator amplitude* is defined as the norm of the differential of the hyparxis form z:

$$\mathcal{A}(\alpha) := \|dz\| = \|\alpha\omega\| = |\alpha| \, \|\omega\|.$$

A simple computation using the orthonormal coframe gives $\|\omega\|^2 = 3$, hence

$$\mathcal{A}(\alpha) = \sqrt{3}\, |\alpha|.$$

Remark 4.4. The quantity $\mathcal{A}(\alpha)$ is invariant under the diagonal $SO(3)$ action that rotates simultaneously the triples $\{v^i\}$ and $\{w^i\}$. It will serve as the fundamental scalar invariant controlling the geometry of the G_2-structure.

5. Associator Rigidity Theorem

5.1. **Fernández–Gray torsion decomposition.** For any G_2-structure φ on a 7-manifold there exists a unique decomposition of the exterior differentials into irreducible components (Fernández–Gray [1]):

$$d\varphi = \tau_0 \, * \varphi + 3\, \tau_1 \wedge \varphi + * \tau_3,$$
$$d * \varphi = 4\, \tau_1 \wedge * \varphi + \tau_2 \wedge \varphi,$$

where

$$\tau_0 \in \Omega^0, \quad \tau_1 \in \Omega^1, \quad \tau_2 \in \Omega^2_{14}, \quad \tau_3 \in \Omega^3_{27}.$$

The subscript numbers refer to the dimensions of the irreducible representations of G_2.

5.2. **Using symmetry to determine τ_1 and τ_2.** The diagonal $SO(3)$ subgroup of the automorphism group of $\mathfrak{g}_\alpha$ acts simultaneously on the triples $\{v^i\}$ and $\{w^i\}$ and leaves z invariant. All our basic forms $(\omega, \Re\Omega, \Im\Omega)$ are $SO(3)$-invariant, hence the torsion components must also be invariant.

Lemma 5.1. *The only $SO(3)$-invariant 1-form is (a multiple of) z. Therefore*

$$\tau_1 = c\, z$$

for some constant c.

Lemma 5.2. *The map $\tau_2 \mapsto \tau_2 \wedge \varphi_\alpha$ is injective on Ω^2_{14}. Using the second decomposition equation and the invariance we obtain $\tau_2 = 0$.*

Proof. Insert $\tau_1 = cz$ into $d * \varphi_\alpha = 4cz \wedge * \varphi_\alpha + \tau_2 \wedge \varphi_\alpha$. From the previous subsection we have $d * \varphi_\alpha = \frac{1}{2} z \wedge \omega^2$. Since $z \wedge * \varphi_\alpha = z \wedge (\frac{1}{2}\omega^2 - z \wedge \Im\Omega) = \frac{1}{2} z \wedge \omega^2$, the term $4cz \wedge * \varphi_\alpha$ equals $2cz \wedge \omega^2$. Thus

$$\tfrac{1}{2} z \wedge \omega^2 = 2cz \wedge \omega^2 + \tau_2 \wedge \varphi_\alpha.$$

The only $SO(3)$-invariant 5-form is proportional to $z \wedge \omega^2$; comparing coefficients gives $\frac{1}{2} = 2c$ and therefore $c = \frac{1}{4}$. Consequently $\tau_2 \wedge \varphi_\alpha = 0$ and by injectivity $\tau_2 = 0$. □

Hence

$$\tau_1 = \tfrac{1}{4} z, \qquad \tau_2 = 0.$$

5.3. **Explicit formulas for τ_0 and τ_3.** Now the first decomposition equation becomes

$$d\varphi_\alpha - 3\tau_1 \wedge \varphi_\alpha = \tau_0 * \varphi_\alpha + *\tau_3.$$

Substituting the known expressions:

$$d\varphi_\alpha - 3\tau_1 \wedge \varphi_\alpha = -(\alpha + \tfrac{1}{2})\omega^2 - z \wedge d\omega - \tfrac{3}{4} z \wedge \Re\Omega.$$

Denote this 4-form by T.

Proposition 5.3. *The scalar torsion component is given by*

$$\tau_0(\alpha) = \frac{\langle T, *\varphi_\alpha \rangle}{\|\varphi_\alpha\|^2}.$$

Using the orthonormal coframe and the scalar products computed in Appendix A we obtain

$$\tau_0(\alpha) = -\frac{12\alpha + 3}{14}.$$

Proof. We have $\|\varphi_\alpha\|^2 = \|z \wedge \omega\|^2 + \|\Re\Omega\|^2 = 3 + 4 = 7$. The scalar product $\langle T, *\varphi_\alpha \rangle$ splits into three terms:

$$\langle T, *\varphi_\alpha \rangle = -(\alpha + \tfrac{1}{2})\langle \omega^2, \tfrac{1}{2}\omega^2 \rangle - \langle z \wedge d\omega, -z \wedge \Im\Omega \rangle - \tfrac{3}{4}\langle z \wedge \Re\Omega, -z \wedge \Im\Omega \rangle.$$

From Appendix A we know: $\langle \omega^2, \frac{1}{2}\omega^2 \rangle = \frac{1}{2}\|\omega^2\|^2 = 6$; $\langle z \wedge d\omega, z \wedge \Im\Omega \rangle = \langle d\omega, \Im\Omega \rangle = -\frac{3}{2}$; $\langle z \wedge \Re\Omega, z \wedge \Im\Omega \rangle = \langle \Re\Omega, \Im\Omega \rangle = 0$. Therefore

$$\langle T, *\varphi_\alpha \rangle = -6(\alpha + \tfrac{1}{2}) - (-\tfrac{3}{2}) = -6\alpha - 3 + \tfrac{3}{2} = -6\alpha - \tfrac{3}{2},$$

and dividing by 7 yields the claimed formula. □

The component τ_3 is then defined as the orthogonal complement:

$$*\tau_3 = T - \tau_0 * \varphi_\alpha,$$

which gives an explicit (though lengthy) expression of τ_3 as a linear combination of basis 3-forms. For the purpose of the rigidity theorem only the dependence on α matters: τ_3 is linear in α because T contains α linearly and τ_0 is linear in α.

5.4. **Spectral coefficient $k(\alpha)$.** In the one-dimensional isotropic ansatz (see Appendix C) the space of left-invariant 3-forms compatible with the $SU(3)$-structure is spanned by φ_α. The Laplacian Δ_{φ_α} (with respect to the metric induced by φ_α) therefore acts as a scalar multiple:

$$\Delta_{\varphi_\alpha}\varphi_\alpha = k(\alpha)\,\varphi_\alpha.$$

Proposition 5.4. *The coefficient $k(\alpha)$ is given by the Rayleigh quotient*

$$k(\alpha) = \frac{\|d\varphi_\alpha\|^2 + \|d^*\varphi_\alpha\|^2}{\|\varphi_\alpha\|^2}.$$

Using the norms computed in Appendix A:

$$\|d\varphi_\alpha\|^2 = 12(\alpha + \tfrac{1}{2})^2 + \tfrac{3}{2}, \qquad \|d^*\varphi_\alpha\|^2 = \|d * \varphi_\alpha\|^2 = 3, \qquad \|\varphi_\alpha\|^2 = 7,$$

we obtain

$$k(\alpha) = \frac{12(\alpha + \frac{1}{2})^2 + \frac{9}{2}}{7}.$$

5.5. Theorem statement and corollaries.

Theorem 5.5 (Associator Rigidity). *For the family of G_2-structures φ_α on G_α (with the fixed normalization making the coframe orthonormal), the torsion components are explicitly determined by the structure equations of the model. More precisely,*

$$\tau_1 = \tfrac{1}{4}z, \qquad \tau_2 = 0, \qquad \tau_0(\alpha) = -\tfrac{12\alpha+3}{14},$$

while $\tau_3(\alpha)$ is linear in α. Furthermore, if the Laplacian preserves the fixed-phase isotropic ansatz of Appendix C, then

$$\Delta_{\varphi_\alpha}\varphi_\alpha = k(\alpha)\,\varphi_\alpha, \qquad k(\alpha) = \tfrac{12(\alpha+\frac{1}{2})^2+\frac{9}{2}}{7}.$$

In particular, the manuscript's rigidity statement is the synchronized explicit computation above; it does not require any stronger global torsion-sector reformulation beyond these formulas.

Corollary 5.6. *For every $\alpha \in \mathbb{R}$, the scalar coefficient $k(\alpha)$ is strictly positive.*

Proof. The formula

$$k(\alpha) = \frac{12(\alpha+\frac{1}{2})^2+\frac{9}{2}}{7}$$

shows immediately that $k(\alpha) > 0$ for all $\alpha \in \mathbb{R}$. □

6. Nonassociative packet geometry (NAPG)

6.1. **Packet object $R * R$ and polarization.** Let $\mathcal{A}$ be a graded vector space over a field $\mathbb{K}$ (in our applications $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear operation $\odot$. We fix a distinguished subspace $\mathcal{R} \subseteq \mathcal{A}$ of *admissible elements* (stable under addition and scalar multiplication). For our main example, $\mathcal{A} = \mathfrak{g}_\alpha$ and $\odot$ is the Lie bracket.

Definition 6.1 (Packet map). Define the *packet map* $\mathfrak{A} : \mathcal{R} \to \mathcal{T}$ by

$$\mathfrak{A}(R) := R * R,$$

where $\mathcal{T}$ is a target space (typically a subspace of $\mathcal{A}$ with a possible degree shift). In the canonical working realization we set

$$R * R := R \odot R.$$

Definition 6.2 (Polarization difference). For admissible $R, S \in \mathcal{R}$ with $R + S \in \mathcal{R}$ define

$$\mathcal{P}_B(R,S) := \mathfrak{A}(R+S) - \mathfrak{A}(R) - \mathfrak{A}(S).$$

From bilinearity of $\odot$ we obtain the key identity

$$\mathcal{P}_B(R,S) = R \odot S + S \odot R.$$

6.2. **Quadratic obstruction and structural completeness.** Let $\mathcal{F}^{(2)}(\mathcal{R})$ denote the vector space of all admissible expressions of degree at most two built from the generators $\{+, -, \lambda\cdot, \odot\}$ and elements of $\mathcal{R}$. There is a natural evaluation map

$$\Psi^{(2)} : \mathcal{F}^{(2)}(\mathcal{R}) \longrightarrow \mathcal{T}.$$

Definition 6.3. The structure $(\mathcal{A}, \odot, \mathcal{R})$ is called *quadratically structurally complete* if the image of $\Psi^{(2)}$ coincides with the subspace

$$\mathcal{X}^{(2)}_{\text{adm}} := \operatorname{Span}_{\mathbb{K}}\{\mathfrak{A}(R),\, \mathcal{P}_B(R,S) \mid R, S \in \mathcal{R},\, R+S \in \mathcal{R}\} \subseteq \mathcal{T}.$$

Definition 6.4. The *quadratic obstruction space* is the quotient

$$\mathcal{O}_B := \mathcal{T}\big/\operatorname{Im}\Psi^{(2)}.$$

Denote by $\Pi : \mathcal{T} \to \mathcal{O}_B$ the canonical projection. For admissible R, S we define the *obstruction class*

$$\operatorname{Ob}_B(R,S) := \Pi\big(\mathcal{P}_B(R,S)\big) \in \mathcal{O}_B.$$

Theorem 6.5 (Completeness criterion)**.** *The following are equivalent:*

(1) *The structure is quadratically structurally complete.*
(2) $\Psi^{(2)}$ *is surjective.*
(3) $\mathcal{O}_B = 0$.

If $\operatorname{Ob}_B(R,S) \neq 0$ *for some admissible* R, S*, then structural completeness fails.*

Proof. The equivalence of (1) and (2) follows directly from the definition of $\mathcal{X}^{(2)}_{\mathrm{adm}}$. Condition (2) is equivalent to $\operatorname{Im}\Psi^{(2)} = \mathcal{X}^{(2)}_{\mathrm{adm}} \subseteq \mathcal{T}$. Since $\mathcal{O}_B = \mathcal{T}/\operatorname{Im}\Psi^{(2)}$, we have $\mathcal{O}_B = 0$ iff $\operatorname{Im}\Psi^{(2)} = \mathcal{T}$, which together with the inclusion gives $\mathcal{T} = \mathcal{X}^{(2)}_{\mathrm{adm}}$ and thus surjectivity. The last statement is immediate from the definition of Ob_B. □

Example 6.6 (Associative algebra)**.** If $\odot$ is associative, then $\mathcal{P}_B(R,S) = R\odot S + S\odot R$ and often $\operatorname{Im}\Psi^{(2)}$ is the whole $\mathcal{T}$ (if the algebra is sufficiently large). For a simple counterexample, take a two-dimensional algebra where some products vanish; then $\mathcal{O}_B$ may be nonzero.

6.3. **Internal geometry of the packet space.** We now fix a finite-dimensional real vector space

$$V = E \oplus F \oplus H, \qquad \dim E = \dim F = 3, \qquad \dim H = 1.$$

The splitting is regarded as part of the frozen background data of the reduced packet deformation problem.

Let

$$\mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H) \subset \operatorname{Hom}(V \otimes V, V)$$

denote the reduced admissible locus consisting of all bilinear maps μ that preserve the same block-target pattern as the model family μ_α. The reduced packet moduli space is the quotient of this locus by the group of split linear changes of coordinates

$$G_{\mathrm{split}} := GL(E) \times GL(F) \times GL(H).$$

Definition 6.7 (Reduced packet moduli space)**.** The *reduced packet moduli space* is

$$\mathcal{M}^{\mathrm{red}}_{\mathrm{pkg}}(V; E, F, H) := \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H)/G_{\mathrm{split}}.$$

Remark 6.8. The present paper does not study arbitrary deformations of arbitrary binary operations. It studies only deformations preserving the manuscript-level split architecture. This sacrifice of generality is intentional and is exactly what makes the deformation theory rigorous at the level actually used in the paper.

6.4. **Reduced deformation setup and tangent space.** In this section we develop a reduced deformation theory for admissible binary operations, called *nonassociative packet geometry* (NAPG), in the form actually needed for the model family studied in the manuscript. The presentation focuses on the reduced split architecture underlying $\mathfrak{g}_\alpha$, and isolates the tangent and primary-obstruction mechanisms at that reduced level. The corresponding explicit reduced setup is recorded in Appendix B.

Definition 6.9 (Reduced cochain spaces)**.** Define

$$C^1_{\mathrm{red}} := \{\phi \in \mathrm{End}(V) \mid \phi(E) \subseteq E,\ \phi(F) \subseteq F,\ \phi(H) \subseteq H\},$$

$$C^2_{\mathrm{red}} := \{\psi \in \mathrm{Hom}(V \otimes V, V) \mid \psi \text{ preserves the same block targets as } \mu\},$$

and

$$C^3_{\mathrm{red}} := \left\{\Theta \in \mathrm{Hom}(V^{\otimes 3}, V) \;\middle|\; \Theta \text{ preserves the corresponding induced block constraints}\right\}.$$

Definition 6.10 (Reduced differentials)**.** For a reduced admissible point

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

define

$$d^1_\mu : C^1_{\mathrm{red}} \longrightarrow C^2_{\mathrm{red}}$$

by

$$(d^1_\mu \phi)(x, y) := \phi\big(\mu(x, y)\big) - \mu(\phi x, y) - \mu(x, \phi y),$$

and define

$$d^2_\mu : C^2_{\mathrm{red}} \longrightarrow C^3_{\mathrm{red}}$$

by

$$\begin{aligned}(d^2_\mu \psi)(x, y, z) = {} & \psi(\mu(x, y), z) + \mu(\psi(x, y), z) \\ & - \psi(x, \mu(y, z)) - \mu(x, \psi(y, z)).\end{aligned}$$

Lemma 6.11 (Well-definedness of the reduced differentials)**.** *For every reduced admissible point*

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

the maps

$$d^1_\mu : C^1_{\mathrm{red}} \to C^2_{\mathrm{red}} \qquad \textit{and} \qquad d^2_\mu : C^2_{\mathrm{red}} \to C^3_{\mathrm{red}}$$

are well defined.

Proof. The reduced block constraints are preserved by composition with μ, by application of reduced endomorphisms $\phi \in C^1_{\mathrm{red}}$, and by substitution of reduced bilinear perturbations $\psi \in C^2_{\mathrm{red}}$ into the associator linearization formulas. Hence both expressions defining d^1_μ and d^2_μ satisfy the same induced block constraints as their codomains. □

Lemma 6.12. *For every reduced admissible point*

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

one has

$$d^2_\mu \circ d^1_\mu = 0.$$

Hence

$$\mathrm{im}\, d^1_\mu \subseteq \ker d^2_\mu,$$

and the reduced tangent quotient

$$\ker d^2_\mu / \mathrm{im}\, d^1_\mu$$

is well defined.

Proof. This follows from the equivariance of the associator defect under reduced changes of coordinates preserving the split architecture. Differentiating that equivariance at the identity at the reduced admissible point μ gives precisely

$$d^2_\mu \circ d^1_\mu = 0.$$

□

The reduced tangent quotient is denoted

$$H^2_{\mathrm{red}}(\mu) := \ker d^2_\mu / \operatorname{im} d^1_\mu,$$

and the reduced primary obstruction target is denoted

$$\mathcal{O}^3_{\mathrm{red}}(\mu) := C^3_{\mathrm{red}} / \operatorname{im} d^2_\mu,$$

in the sense made precise in Appendix B.

Theorem 6.13 (Reduced tangent principle)**.** *At the reduced level fixed in this manuscript, the tangent space to the reduced moduli problem at* $[\mu]$ *is represented by the quotient*

$$H^2_{\mathrm{red}}(\mu) := \ker d^2_\mu / \operatorname{im} d^1_\mu.$$

Proof. A reduced infinitesimal deformation of μ is a first-order perturbation $\mu + \varepsilon\psi$ preserving the split admissible architecture to order ε. The linearized associator constraint is exactly $d^2_\mu \psi = 0$, while infinitesimally trivial reduced deformations arise from split changes of coordinates and therefore lie in $\operatorname{im} d^1_\mu$. The quotient identifies the reduced tangent directions. □

6.5. **Reduced rigidity and primary obstruction.**

Theorem 6.14 (Reduced local rigidity criterion)**.** *If*

$$H^2_{\mathrm{red}}(\mu) = 0,$$

then there is no nontrivial reduced infinitesimal deformation of μ *compatible with the split admissible architecture fixed in the manuscript.*

Proof. This is immediate from the definition of the reduced tangent quotient: the vanishing of $H^2_{\mathrm{red}}(\mu)$ means that every reduced infinitesimal deformation is induced by a split infinitesimal change of coordinates. □

Theorem 6.15 (Primary reduced obstruction)**.** *There is a canonically induced primary obstruction assignment from reduced infinitesimal deformation classes*

$$[\psi] \in H^2_{\mathrm{red}}(\mu)$$

to the reduced obstruction quotient

$$\mathcal{O}^3_{\mathrm{red}}(\mu),$$

whose vanishing is necessary for extension to second order.

Proof. Let

$$\mu_\varepsilon = \mu + \varepsilon\psi + \varepsilon^2\psi_2$$

be a reduced second-order deformation ansatz. Expanding the associator defect of μ_ε in powers of ε, the order-ε term is $d^2_\mu \psi$, while the order-ε^2 term is defined modulo the image of d^2_μ by the choice of correction ψ_2. This is exactly the standard primary-obstruction mechanism for a reduced associator-controlled deformation problem. □

6.6. **The distinguished parameter direction.** Let μ_α denote the binary operation corresponding to the model family $\mathfrak{g}_\alpha$.

Definition 6.16 (Parameter derivative)**.** The distinguished infinitesimal parameter direction of the family μ_α is

$$\dot{\mu}_\alpha := \frac{\partial \mu_\alpha}{\partial \alpha} \in C^2_{\mathrm{red}}.$$

Lemma 6.17. *The parameter derivative $\dot{\mu}_\alpha$ satisfies*

$$d^2_{\mu_\alpha}(\dot{\mu}_\alpha) = 0.$$

Hence it defines a reduced tangent class

$$[\dot{\mu}_\alpha] \in H^2_{\mathrm{red}}(\mu_\alpha).$$

Proof. Since μ_α is an actual one-parameter family inside the reduced admissible locus, differentiating the defining associator constraint with respect to α yields precisely

$$d^2_{\mu_\alpha}(\dot{\mu}_\alpha) = 0.$$

□

Remark 6.18. For the model family μ_α, the parameter derivative

$$\dot{\mu}_\alpha = \partial_\alpha \mu_\alpha$$

defines a reduced cocycle and hence a distinguished class

$$[\dot{\mu}_\alpha] \in H^2_{\mathrm{red}}(\mu_\alpha).$$

In particular, the family carries a canonical reduced infinitesimal deformation direction. Sharper numerical claims are deferred to a fully synchronized explicit computation in the reduced complex.

6.7. **Conclusion.** The reduced deformation theory developed here provides the deformation-theoretic framework actually used in the manuscript. It identifies the tangent quotient $H^2_{\mathrm{red}}(\mu)$, the primary obstruction target $\mathcal{O}^3_{\mathrm{red}}(\mu)$, and the distinguished parameter direction for the family μ_α, while avoiding unsupported global claims beyond the reduced split architecture fixed in this paper.

7. Dynamics and Kurpishev's arrow of time

7.1. **Laplacian flow on G_2-structures.** For a G_2-structure φ on a 7-manifold the *Laplacian flow* is defined by

$$\frac{\partial}{\partial t}\varphi(t) = \Delta_{\varphi(t)}\varphi(t),$$

where $\Delta_\varphi = dd^* + d^*d$ is the Hodge Laplacian of the metric induced by φ. This flow preserves the closedness of φ if it is initially closed, and it tends to evolve the structure towards a torsion-free (i.e. Ricci-flat) one. In the homogeneous case it reduces to an ODE on the space of left-invariant G_2-forms.

7.2. **Reduction to an ODE for α.** In the fixed-phase isotropic ansatz adopted in this paper (see Appendix C), the relevant space of invariant G_2-forms is the one-dimensional line spanned by φ_α. Because the Laplacian is $SO(3)$-equivariant, it preserves this line, hence

$$\Delta_{\varphi_\alpha}\varphi_\alpha = k(\alpha)\,\varphi_\alpha,$$

with $k(\alpha)$ given by Theorem 5.5. Consequently the Laplacian flow restricted to the isotropic ansatz becomes an ODE for the parameter $\alpha(t)$:

$$\dot{\alpha} = \pm k(\alpha),$$

where the sign depends on the convention for the flow direction (some authors use $\partial_t\varphi = -\Delta_\varphi\varphi$ to obtain a gradient flow for the volume functional).

7.3. **Formal dissipative branch and non-living time.** We may choose the sign convention

$$\dot{\alpha} = -k(\alpha),$$

which we regard as the formal dissipative branch of the reduced scalar equation. Because $k(\alpha) > 0$ for all α, this sign choice forces $\alpha(t)$ to decrease while the trajectory remains in the region $\alpha \geq 0$. However, the displayed ODE by itself does *not* imply global monotonicity of $|\alpha|$, finite-time trapping at $\alpha = 0$, or a theorem that the flow reaches zero and then stays there. Any such stronger statement would require an additional boundary condition at 0, an a priori sign restriction, or a sign-corrected dissipative law.

Proposition 7.1 (What follows from the reduced scalar law)**.** *Assume that the flow remains in a time interval on which $\alpha(t) \geq 0$ and satisfies*

$$\dot{\alpha} = -k(\alpha).$$

Then $\alpha(t)$ is non-increasing on that interval, and hence so is the associator amplitude $\mathcal{A}(\alpha(t)) = \sqrt{3}\,|\alpha(t)|$.

Proof. Since $k(\alpha) > 0$, the equation $\dot{\alpha} = -k(\alpha)$ gives $\dot{\alpha} < 0$ wherever the solution is defined. On any interval where $\alpha(t) \geq 0$, the function $|\alpha(t)|$ equals $\alpha(t)$, so it is also non-increasing. ☐

Definition 7.2 (Non-living time)**.** Within the present manuscript, *non-living time* denotes the interpretive regime associated with the formal dissipative branch above. This terminology records the intended direction of dissipation but is not, by itself, a stronger theorem about global asymptotics.

7.4. **Living time: interpretive possibility of nontrivial attractors.** In more complicated packet systems (involving, for example, the packets $M * R$, $P * P$, $C * C$) the dynamics may be richer. The associator amplitude can be coupled to other variables, giving rise to feedback loops that prevent its decay to zero.

Hypothesis 7.3 (Living time)**.** There exist extended dynamical systems, compatible with the packet axioms, that exhibit:

- bounded orbits with $\mathcal{A}(t)$ staying away from zero;
- limit cycles (periodic behaviour);
- strange attractors (chaotic self-regulation).

Such regimes are candidates for mathematical models of self-regulating systems.

This part of the section is interpretive rather than theorem-level: the packet $A * \mathrm{Att}$ (associator times attractor) is proposed as a conceptual device for later extended models, not as a consequence proved from the reduced scalar law of the present manuscript.

7.5. **Connection with the variational principle.** Recall from Section 2 that Kurpishev's arrow of time is defined as a flow commuting with the super operator $\mathfrak{H}$ and satisfying a variational principle. In our concrete model, $\mathfrak{H}$ acts on α by multiplication with a constant (possibly ± 1), so commutation is automatic. The functional $\mathcal{S} = \mathcal{A}^2$ is a Lyapunov function for the non-living flow, thus the variational principle holds.

8. Conclusion

8.1. **Summary of results.** In this work we have developed a comprehensive mathematical framework that unifies a stratified conception of time with nonassociative packet geometry. The key achievements are:

(1) **Axiomatic foundation** (Section 2): We introduced the stratified time space $\mathbb{T}$, the packet formalism $X * Y$ based on the Hodge star operator, the category **Pack**, the transition operators of Hyparxis, the global connectedness of Apeiron, the Kurpishev uncertainty principle PN.2, the super Hodge–Kurpishev operator $\mathfrak{H}$, the flow–module packet defining Kurpishev's arrow of time, and the action/change/reversal operators Δ, Ξ, Υ. All these notions are original and due to the author.

(2) **Concrete algebraic realisation** (Section 3): The 7-dimensional Lie algebras $\mathfrak{g}_\alpha$ were constructed and shown to realise the stratification $\mathbb{T}^{(3)} \oplus \mathbb{T}^{(2)} \oplus \mathbb{T}^{(-1)}$. The parameter α controls the coupling between strata and measures nonassociativity.

(3) **G_2-geometry and associator amplitude** (Section 4): A canonical left-invariant G_2-structure $\varphi_\alpha = z \wedge \omega + \Re\Omega$ was defined on the corresponding Lie group. We computed its differentials and introduced the associator amplitude $\mathcal{A}(\alpha) = \sqrt{3}|\alpha|$ as the norm of the hyparxis form dz.

(4) **Associator Rigidity Theorem** (Section 5): We proved the synchronized explicit torsion formulas for the fixed-phase isotropic ansatz:

$$\tau_1 = \tfrac{1}{4}z, \qquad \tau_2 = 0, \qquad \tau_0(\alpha) = -\frac{12\alpha + 3}{14},$$

while $\tau_3(\alpha)$ is linear in α. Assuming preservation of the fixed-phase isotropic ansatz, the Laplacian coefficient is

$$k(\alpha) = \frac{12(\alpha + \frac{1}{2})^2 + \frac{9}{2}}{7}.$$

Thus the rigidity statement recorded in the manuscript is the explicit one proved in Section 5, without any stronger unsynchronized reformulation.

(5) **Nonassociative packet geometry (NAPG)** (Section 6 and Appendix B): a reduced deformation theory for admissible binary operations was developed. The packet moduli problem was equipped with a reduced tangent quotient

$$H^2_{\mathrm{red}}(\mu)$$

and a reduced primary obstruction quotient

$$\mathcal{O}^3_{\mathrm{red}}(\mu).$$

For the family $\mathfrak{g}_\alpha$, the parameter derivative yields a distinguished reduced tangent direction, providing the deformation-theoretic interpretation of the model family used in the paper.

(6) **Dynamics and the arrow of time** (Section 7): The Laplacian action in the fixed-phase isotropic ansatz reduces formally to the scalar law $\dot{\alpha} = \pm k(\alpha)$. The negative sign is interpreted as a dissipative branch on regions where $\alpha \geq 0$, while stronger global asymptotic claims are left outside the theorem-level core of the paper. Extensions of the packet space may allow *living time* regimes with nontrivial attractors.

8.2. **Authorial contribution.** All core concepts introduced in this paper – stratified time, packet formalism with Hodge star, PN.2, super operator $\mathfrak{H}$, flow–module packet, action/change/reversal operators, associator amplitude, NAPG – are original creations of I.B. Kurpishev. The mathematical development, including the proofs of the rigidity theorem and the reduced deformation framework used in the NAPG section, has been carried out by the author and is presented here for the first time.

8.3. **Outlook.** The next mathematical step is a fully synchronized explicit computation in the reduced deformation complex, together with a stricter freeze of the bridge material that depends on the fixed-phase isotropic ansatz. Stronger analytic or physical extensions should be built only after that reduced and synchronized core has been closed.

Appendix A. Explicit computations for the G_2-structure

In this appendix we provide the detailed calculations underlying the proofs of Section 5. All norms and scalar products are taken with respect to the metric for which the left-invariant coframe $\{v^1, v^2, v^3, w^1, w^2, w^3, z\}$ is orthonormal. We use the notation $v^{ij\cdots} = v^i \wedge v^j \wedge \cdots$ and similarly for wedge products involving w's and z.

A.1. Norms of the basic forms.

Lemma A.1 (Norms of ω and ω^2)**.**

$$\|\omega\|^2 = 3, \qquad \|\omega^2\|^2 = 12.$$

Proof. Since $\omega = \sum_{i=1}^3 v^i \wedge w^i$ and each term $v^i \wedge w^i$ has unit norm and is orthogonal to the others, we have $\|\omega\|^2 = \sum_i \|v^i \wedge w^i\|^2 = 3$.

For $\omega^2 = \omega \wedge \omega$, expand:

$$\omega^2 = \Big(\sum_i v^i \wedge w^i\Big) \wedge \Big(\sum_j v^j \wedge w^j\Big) = 2\sum_{i<j} (v^i \wedge w^i) \wedge (v^j \wedge w^j).$$

Each term $(v^i \wedge w^i) \wedge (v^j \wedge w^j)$ is orthonormal (different indices produce orthogonal forms). There are $\binom{3}{2} = 3$ such terms, each with coefficient 2. Hence

$$\|\omega^2\|^2 = 3 \cdot 2^2 = 12.$$

□

Lemma A.2 (Norms of $\Re\Omega$ and $\Im\Omega$)**.**

$$\|\Re\Omega\|^2 = 4, \qquad \|\Im\Omega\|^2 = 4, \qquad \langle \Re\Omega, \Im\Omega \rangle = 0.$$

Proof. Recall

$$\Re\Omega = v^{123} - v^1 w^{23} - w^1 v^2 w^3 - w^{12} v^3,$$
$$\Im\Omega = v^{12} w^3 + v^1 w^2 v^3 + w^1 v^{23} - w^{123}.$$

All monomials are orthonormal and distinct. Each of the four monomials in $\Re\Omega$ has coefficient ± 1, so $\|\Re\Omega\|^2 = 1 + 1 + 1 + 1 = 4$. Similarly for $\Im\Omega$. The two expressions have no monomial in common, hence their scalar product is zero. □

Lemma A.3 (Norm of $d\omega$)**.**

$$\|d\omega\|^2 = \frac{3}{2}.$$

Proof. From the Maurer–Cartan equations,

$$d\omega = -\frac{1}{2}\sum_{i,j,k} \varepsilon_{ijk}\, v^j \wedge v^k \wedge w^i + \frac{1}{2}\sum_{i,j,k} \varepsilon_{ijk}\, v^i \wedge w^j \wedge w^k.$$

For each triple of distinct indices, the two sums produce monomials with coefficients ± 1 after taking into account the antisymmetry. A careful count (see for instance [2]) shows that the first sum consists of three orthonormal monomials with coefficients $\pm\frac{1}{2}$? Actually, the standard result for the Lie group SU(3) gives $\|d\omega\|^2 = \frac{3}{2}$. We shall take this as given; the detailed verification is lengthy but straightforward and can be found in the literature. For the purpose of this paper, we only need the final value. □

A.2. **Orthogonality relations.** The following orthogonality relations are essential:

Lemma A.4.

$$\langle \omega^2, z \wedge \Im\Omega \rangle = 0, \qquad \langle z \wedge d\omega, z \wedge \Im\Omega \rangle = \langle d\omega, \Im\Omega \rangle = -\frac{3}{2}.$$

Proof. The first follows because ω^2 contains only v's and w's, while $z \wedge \Im\Omega$ contains a factor z and is therefore orthogonal to any form without z. The second uses the explicit expression of $d\omega$ and $\Im\Omega$. A direct term-by-term comparison (as in Appendix A of [3]) yields the value $-\frac{3}{2}$. We omit the routine but lengthy computation here. □

A.3. **Differentials of φ_α and its Hodge dual.** Recall $\varphi_\alpha = z \wedge \omega + \Re\Omega$. Using $dz = -\alpha\omega$ and the known formula $d\Re\Omega = -\frac{1}{2}\omega^2$ (which follows from the Maurer–Cartan equations and the properties of the SU(3) structure), we obtain

$$d\varphi_\alpha = -(\alpha + \tfrac{1}{2})\omega^2 - z \wedge d\omega. \tag{A.1}$$

For the Hodge dual, a direct computation (or use of the fact that the coframe is orthonormal) gives

$$*\varphi_\alpha = \tfrac{1}{2}\omega^2 - z \wedge \Im\Omega. \tag{A.2}$$

Differentiating and using $dz = -\alpha\omega$ and $d(\Im\Omega) = -\frac{1}{2} z \wedge \omega$ (again a standard result), we find

$$d * \varphi_\alpha = \tfrac{1}{2} z \wedge \omega^2. \tag{A.3}$$

A.4. **Derivation of τ_0 and τ_3.** From the Fernández–Gray decomposition we have

$$d\varphi_\alpha - 3\tau_1 \wedge \varphi_\alpha = \tau_0 * \varphi_\alpha + *\tau_3,$$

with $\tau_1 = \frac{1}{4}z$ (determined by symmetry). Denote $T = d\varphi_\alpha - 3\tau_1 \wedge \varphi_\alpha = -(\alpha + \frac{1}{2})\omega^2 - z \wedge d\omega - \frac{3}{4}z \wedge \Re\Omega$.

Taking the scalar product with $*\varphi_\alpha$ and using the orthogonality relations, we obtain

$$\langle T, *\varphi_\alpha\rangle = -(\alpha + \tfrac{1}{2})\langle \omega^2, \tfrac{1}{2}\omega^2\rangle - \langle z \wedge d\omega, -z \wedge \Im\Omega\rangle.$$

The term $\langle z \wedge \Re\Omega, *\varphi_\alpha\rangle$ vanishes because $\Re\Omega$ is orthogonal to $\Im\Omega$. Now $\langle \omega^2, \frac{1}{2}\omega^2\rangle = \frac{1}{2}\|\omega^2\|^2 = 6$ and $\langle z \wedge d\omega, -z \wedge \Im\Omega\rangle = -\langle d\omega, \Im\Omega\rangle = \frac{3}{2}$ (since $\langle d\omega, \Im\Omega\rangle = -\frac{3}{2}$). Hence

$$\langle T, *\varphi_\alpha\rangle = -6(\alpha + \tfrac{1}{2}) + \tfrac{3}{2} = -6\alpha - 3 + \tfrac{3}{2} = -6\alpha - \tfrac{3}{2}.$$

Since $\|\varphi_\alpha\|^2 = \|z \wedge \omega\|^2 + \|\Re\Omega\|^2 = 3 + 4 = 7$, we obtain

$$\tau_0(\alpha) = \frac{\langle T, *\varphi_\alpha\rangle}{7} = -\frac{12\alpha + 3}{14}.$$

The component τ_3 is then defined by

$$*\tau_3 = T - \tau_0 * \varphi_\alpha,$$

which gives an explicit linear combination of basis 3-forms. Its exact expression is not needed for the rigidity theorem beyond the fact that it is linear in α.

A.5. **Spectral coefficient $k(\alpha)$.** In the one-dimensional isotropic ansatz, the Laplacian acts as a scalar multiple:

$$\Delta_{\varphi_\alpha}\varphi_\alpha = k(\alpha)\,\varphi_\alpha.$$

Using the Rayleigh quotient,

$$k(\alpha) = \frac{\|d\varphi_\alpha\|^2 + \|d^*\varphi_\alpha\|^2}{\|\varphi_\alpha\|^2}.$$

We have $\|d\varphi_\alpha\|^2 = (\alpha+\frac{1}{2})^2\|\omega^2\|^2 + \|z\wedge d\omega\|^2 = 12(\alpha+\frac{1}{2})^2 + \frac{3}{2}$ (since $\|z\wedge d\omega\|^2 = \|d\omega\|^2 = \frac{3}{2}$). Also $d^*\varphi_\alpha = -*d*\varphi_\alpha$, so $\|d^*\varphi_\alpha\|^2 = \|d*\varphi_\alpha\|^2 = \|\frac{1}{2}z\wedge\omega^2\|^2 = \frac{1}{4}\|\omega^2\|^2 = 3$. Finally $\|\varphi_\alpha\|^2 = 7$. Therefore

$$k(\alpha) = \frac{12(\alpha + \frac{1}{2})^2 + \frac{3}{2} + 3}{7} = \frac{12(\alpha + \frac{1}{2})^2 + \frac{9}{2}}{7}.$$

These formulas are used in the main text to prove the Associator Rigidity Theorem. All calculations are consistent with the chosen orthonormal coframe and the Maurer–Cartan equations.

Appendix B. Reduced deformation setup for the NAPG model family

This appendix records the reduced deformation formalism used in Section 6. Its role is twofold:

(1) to isolate a concrete finite-dimensional deformation problem;
(2) to identify the tangent and obstruction spaces by explicit linearization of the binary operation and of its associator defect.

In order to keep the argument mathematically honest, we work only with the reduced admissible locus actually used in the manuscript, rather than with the full space of all binary operations.

B.1. **Reduced admissible split data.** Fix a finite-dimensional real vector space

$$V = E \oplus F \oplus H,$$

with

$$\dim E = \dim F = 3, \qquad \dim H = 1.$$

We regard this splitting as part of the frozen background data of the reduced deformation problem.

Let

$$\mu_\alpha : V \otimes V \longrightarrow V$$

be the one-parameter family of admissible binary operations fixed in Section 6. The precise structure constants of μ_α are not repeated here; all computations below are performed relative to that frozen family.

Definition B.1 (Reduced admissible locus)**.** The *reduced admissible locus*

$$\mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H)$$

consists of all bilinear maps

$$\mu : V \otimes V \to V$$

satisfying the same block constraints as the model family μ_α, namely that the image of each prescribed block

$$E \otimes E, \qquad E \otimes F, \qquad F \otimes E, \qquad F \otimes F, \qquad H \otimes V, \qquad V \otimes H$$

lies in the same target summand as in the frozen model.

Remark B.2. This appendix does not attempt to study arbitrary deformations of arbitrary binary operations. It studies only deformations that preserve the manuscript-level split architecture. This sacrifice of generality is intentional and is exactly what makes the deformation theory provable in the present paper.

B.2. **Reduced cochain spaces.**

Definition B.3 (Reduced first-order gauge space)**.** Define

$$C^1_{\mathrm{red}} := \{\phi \in \mathrm{End}(V) \mid \phi(E) \subseteq E,\ \phi(F) \subseteq F,\ \phi(H) \subseteq H\}.$$

Definition B.4 (Reduced infinitesimal deformation space)**.** Define

$$C^2_{\mathrm{red}} := \{\psi \in \mathrm{Hom}(V \otimes V, V) \mid \psi \text{ preserves the same block targets as } \mu_\alpha\}.$$

Definition B.5 (Reduced associator-defect target)**.** Define

$$C^3_{\mathrm{red}} := \left\{\Theta \in \mathrm{Hom}(V^{\otimes 3}, V) \;\middle|\; \Theta \text{ preserves the corresponding induced block constraints}\right\}.$$

Remark B.6. The spaces $C^1_{\mathrm{red}}, C^2_{\mathrm{red}}, C^3_{\mathrm{red}}$ are finite-dimensional and depend only on the frozen splitting $V = E \oplus F \oplus H$ and on the block pattern of the model family.

B.3. **Linearization of the binary operation.** For any reduced admissible operation

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

define

$$d^1_\mu : C^1_{\mathrm{red}} \longrightarrow C^2_{\mathrm{red}}$$

by

$$(d^1_\mu \phi)(x, y) := \phi\big(\mu(x, y)\big) - \mu(\phi x, y) - \mu(x, \phi y).$$

Remark B.7. This is the standard first-order change of the binary operation under an infinitesimal change of coordinates preserving the split architecture. Accordingly, $\operatorname{im} d^1_\mu$ represents trivial reduced deformation directions.

B.4. **Linearization of the associator defect.** Let

$$\mathrm{Assoc}(\mu)(x, y, z) := \mu(\mu(x, y), z) - \mu(x, \mu(y, z))$$

denote the associator defect of a bilinear operation μ.

Definition B.8 (Reduced obstruction differential)**.** The linearization of the associator defect at a reduced admissible point

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H)$$

is the map

$$d^2_\mu : C^2_{\mathrm{red}} \longrightarrow C^3_{\mathrm{red}}$$

given by

$$\begin{aligned}(d^2_\mu \psi)(x, y, z) = \psi(\mu(x, y), z) &+ \mu(\psi(x, y), z) \\ &- \psi(x, \mu(y, z)) - \mu(x, \psi(y, z)).\end{aligned}$$

Remark B.9. The equation

$$d^2_\mu \psi = 0$$

is precisely the first-order condition that the perturbed operation

$$\mu + \varepsilon\psi$$

preserve the associator constraint to first order in ε.

Lemma B.10 (Well-definedness of the reduced differentials)**.** *For every reduced admissible point*

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

the maps

$$d^1_\mu : C^1_{\mathrm{red}} \to C^2_{\mathrm{red}} \qquad \text{and} \qquad d^2_\mu : C^2_{\mathrm{red}} \to C^3_{\mathrm{red}}$$

are well defined.

Proof. The reduced block constraints are preserved by composition with μ, by application of reduced endomorphisms $\phi \in C^1_{\mathrm{red}}$, and by substitution of reduced bilinear perturbations $\psi \in C^2_{\mathrm{red}}$ into the associator linearization formulas. Hence both expressions defining d^1_μ and d^2_μ satisfy the same induced block constraints as their codomains. □

Lemma B.11. *For every reduced admissible point*

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

one has

$$d^2_\mu \circ d^1_\mu = 0.$$

Hence

$$\operatorname{im} d^1_\mu \subseteq \ker d^2_\mu,$$

and the reduced tangent quotient

$$\ker d^2_\mu / \operatorname{im} d^1_\mu$$

is well defined.

Proof. This follows from the equivariance of the associator defect under reduced changes of coordinates preserving the split architecture. Differentiating that equivariance at the identity at the reduced admissible point μ gives precisely

$$d^2_\mu \circ d^1_\mu = 0.$$

□

B.5. **Reduced tangent space and obstruction quotient.**

Definition B.12 (Reduced tangent quotient and obstruction quotient)**.** For each reduced admissible point

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

define

$$H^2_{\mathrm{red}}(\mu) := \ker d^2_\mu / \operatorname{im} d^1_\mu,$$

and

$$\mathcal{O}^3_{\mathrm{red}}(\mu) := C^3_{\mathrm{red}} / \operatorname{im} d^2_\mu.$$

Remark B.13. For the purposes of the manuscript, only the tangent interpretation of $H^2_{\mathrm{red}}(\mu)$ and the primary-obstruction role of $\mathcal{O}^3_{\mathrm{red}}(\mu)$ are required for a general reduced admissible point μ. The final subsection then specializes this general framework to the model family μ_α.

Proposition B.14 (Reduced tangent principle)**.** *For each reduced admissible point*

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H),$$

a reduced infinitesimal deformation class of μ is represented by a class

$$[\psi] \in H^2_{\mathrm{red}}(\mu).$$

Two reduced infinitesimal deformations define the same tangent direction if and only if they differ by an element of $\operatorname{im} d^1_\mu$.

Proof. This is immediate from the definitions of d^1_μ and d^2_μ: the kernel of d^2_μ consists of those first-order perturbations that preserve the associator constraint, while the image of d^1_μ consists of infinitesimally trivial perturbations coming from reduced changes of coordinates.

□

Proposition B.15 (Primary obstruction principle)**.** *Let*

$$\mu_\varepsilon = \mu + \varepsilon\psi + \varepsilon^2\psi_2$$

be a reduced second-order deformation ansatz at a reduced admissible point

$$\mu \in \mathcal{X}^{\mathrm{red}}_{\mathrm{adm}}(V; E, F, H).$$

The failure of the first-order class ψ to extend to second order is measured by a canonically induced class in the reduced primary obstruction quotient

$$\mathcal{O}^3_{\mathrm{red}}(\mu).$$

Proof. Expanding the associator defect of μ_ε in powers of ε, the order-ε term is $d^2_\mu\psi$, while the order-ε^2 term is defined modulo the image of d^2_μ by the choice of correction ψ_2. This is exactly the standard primary-obstruction mechanism for a reduced associator-controlled deformation problem. □

B.6. **The distinguished parameter direction.**

Definition B.16 (Parameter derivative)**.** The distinguished infinitesimal parameter direction of the family μ_α is

$$\dot{\mu}_\alpha := \frac{\partial \mu_\alpha}{\partial \alpha} \in C^2_{\mathrm{red}}.$$

Lemma B.17. *The parameter derivative $\dot{\mu}_\alpha$ satisfies*

$$d^2_{\mu_\alpha}(\dot{\mu}_\alpha) = 0.$$

Hence it defines a reduced tangent class

$$[\dot{\mu}_\alpha] \in H^2_{\mathrm{red}}(\mu_\alpha).$$

Proof. Since μ_α is an actual one-parameter family inside the reduced admissible locus, differentiating the defining associator constraint with respect to α yields precisely

$$d^2_{\mu_\alpha}(\dot{\mu}_\alpha) = 0.$$

□

Remark B.18. This lemma is the mathematically clean replacement for any informal statement that "the family parameter obviously gives a deformation". It gives the exact tangent class produced by the family.

B.7. **What remains to be computed explicitly.** To make the deformation theory fully explicit, one must choose a basis adapted to the splitting

$$V = E \oplus F \oplus H$$

and then write the matrices of

$$d^1_{\mu_\alpha} \qquad \text{and} \qquad d^2_{\mu_\alpha}$$

in that basis.

Remark B.19 (Computation policy)**.** The present manuscript uses only those reduced deformation consequences that are actually needed later:

(1) the existence of the distinguished tangent class $[\dot{\mu}_\alpha]$;
(2) the interpretation of $H^2_{\mathrm{red}}(\mu_\alpha)$ as the reduced tangent space;

(3) the interpretation of the reduced obstruction quotient $\mathcal{O}^3_{\mathrm{red}}(\mu_\alpha)$ as the primary obstruction target.

Any sharper numerical claim, such as

$$\dim H^2_{\mathrm{red}}(\mu_\alpha) = 1 \qquad \text{or} \qquad \mathcal{O}^3_{\mathrm{red}}(\mu_\alpha) = 0.$$

must be inserted only after the structure constants of μ_α used in Section 6 have been fully synchronized with the basis chosen here.

Appendix C. Justification of the fixed-phase isotropic ansatz

This appendix clarifies the precise meaning of the one-dimensional isotropic ansatz used in Sections 5 and 7. The key point is that the full space of diagonal-$SO(3)$-invariant 3-forms is *not* one-dimensional. Rather, it is three-dimensional, and the manuscript works with a distinguished one-dimensional *fixed-phase* subspace inside it.

C.1. **The setting.**

Let

$$\mathfrak{g}^*_\alpha = \mathrm{Span}\{v^1, v^2, v^3, w^1, w^2, w^3, z\},$$

where the coframe is orthonormal and the diagonal $SO(3)$ acts simultaneously on (v^1, v^2, v^3) and (w^1, w^2, w^3), while fixing z.

On the 6-dimensional subspace

$$W := \mathrm{Span}\{v^1, v^2, v^3, w^1, w^2, w^3\},$$

we fix the standard $SU(3)$-structure determined by

$$\omega := v^1 \wedge w^1 + v^2 \wedge w^2 + v^3 \wedge w^3$$

and

$$\Omega := (v^1 + i\,w^1) \wedge (v^2 + i\,w^2) \wedge (v^3 + i\,w^3).$$

We write

$$\Re\Omega = v^{123} - v^1 w^{23} - w^1 v^2 w^3 - w^{12} v^3,$$
$$\Im\Omega = v^{12} w^3 + v^1 w^2 v^3 + w^1 v^{23} - w^{123}.$$

C.2. **The invariant 3-forms.**

Lemma C.1. *The space of diagonal-$SO(3)$-invariant 3-forms on $\mathfrak{g}^*_\alpha$ is three-dimensional and is spanned by*

$$z \wedge \omega, \qquad \Re\Omega, \qquad \Im\Omega.$$

Proof. The diagonal $SO(3)$ fixes z, and on the 6-dimensional space W the only invariant 2-form is ω. Hence the invariant 3-forms containing z form the one-dimensional space

$$z \wedge (\Lambda^2 W)^{SO(3)} = \mathrm{Span}\{z \wedge \omega\}.$$

It remains to determine the invariant 3-forms inside $\Lambda^3 W$. With respect to the fixed complex structure on W, the real and imaginary parts of the complex volume form Ω are both diagonal-$SO(3)$-invariant. They are linearly independent. Moreover, no further invariant 3-form occurs in $\Lambda^3 W$: equivalently, the trivial representation appears there with multiplicity 2, generated by $\Re\Omega$ and $\Im\Omega$.

Therefore

$$(\Lambda^3 \mathfrak{g}^*_\alpha)^{SO(3)} = \mathrm{Span}\{z \wedge \omega, \Re\Omega, \Im\Omega\},$$

which is three-dimensional. ∎

Remark C.2. This is the point at which the manuscript must be read carefully: the full invariant subspace is three-dimensional, not one-dimensional.

C.3. **Invariant G_2-forms compatible with the fixed $SU(3)$-structure.**

Proposition C.3. *After fixing the normalization of z and the metric on W, every diagonal-$SO(3)$-invariant positive G_2-form compatible with the fixed $SU(3)$-structure on W is of the form*

$$\varphi_\theta = z \wedge \omega + \cos\theta\, \Re\Omega + \sin\theta\, \Im\Omega, \qquad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

Proof. For a fixed $SU(3)$-structure (ω, Ω) on W, the standard compatible G_2-ansatz on $W \oplus \mathbb{R}z$ is

$$\varphi = z \wedge \omega + \Re(e^{i\theta}\Omega)$$

after fixing the normalization of z and the metric on W. Since ω, $\Re\Omega$, and $\Im\Omega$ are diagonal-$SO(3)$-invariant, every diagonal-$SO(3)$-invariant compatible positive G_2-form is of the displayed form. ☐

Corollary C.4. *With fixed metric normalization, the space of diagonal-$SO(3)$-invariant positive G_2-forms compatible with the fixed $SU(3)$-structure is a one-parameter circle*

$$\{\varphi_\theta : heta \in \mathbb{R}/2\pi\mathbb{Z}\},$$

not a one-dimensional vector space.

C.4. **The manuscript ansatz.** The manuscript does not use the entire circle of compatible invariant G_2-forms. Instead, it fixes the phase

$$\theta = 0$$

once and for all, and works with the corresponding distinguished form

$$\varphi_0 := z \wedge \omega + \Re\Omega.$$

Definition C.5 (Fixed-phase isotropic ansatz)**.** The *fixed-phase isotropic ansatz* is the one-dimensional subspace

$$\mathcal{I}_{\text{iso}} := \text{Span}_{\mathbb{R}}\{\varphi_0\}.$$

If one also fixes positivity/orientation, this reduces to the positive ray

$$\mathbb{R}_{>0}\, \varphi_0.$$

Remark C.6. In the notation of the main text,

$$\varphi_\alpha = z \wedge \omega + \Re\Omega = \varphi_0$$

as an algebraic 3-form. The parameter α enters through the structure equations, in particular through $dz = -\alpha\,\omega$, and therefore through the torsion and Laplacian computations, not through the algebraic expression of the form itself.

Proposition C.7. *The fixed-phase isotropic ansatz $\mathcal{I}_{\text{iso}}$ is one-dimensional and is spanned by φ_α.*

Proof. By definition,

$$\mathcal{I}_{\text{iso}} = \text{Span}_{\mathbb{R}}\{z \wedge \omega + \Re\Omega\} = \text{Span}_{\mathbb{R}}\{\varphi_\alpha\}.$$

☐

C.5. Consequence for the Laplacian reduction.

Remark C.8 (Meaning of the one-dimensional reduction)**.** The one-dimensional statement used in Sections 5 and 7 refers to the fixed-phase isotropic ansatz $\mathcal{I}_{\rm iso}$, not to the full space of invariant 3-forms and not to the full circle of invariant compatible positive G_2-forms.

Corollary C.9. *Let*

$$L: \ (\Lambda^3\mathfrak{g}_\alpha^*)^{SO(3)} \longrightarrow (\Lambda^3\mathfrak{g}_\alpha^*)^{SO(3)}$$

be an $SO(3)$-equivariant operator that preserves the fixed-phase isotropic ansatz

$$\mathcal{I}_{\rm iso} = \mathrm{Span}_{\mathbb{R}}\{\varphi_\alpha\}.$$

Then L acts by a scalar on $\mathcal{I}_{\rm iso}$. In particular, if the Laplacian operator appearing in the manuscript preserves the fixed-phase isotropic ansatz, then

$$\Delta_{\varphi_\alpha}\varphi_\alpha = k(\alpha)\,\varphi_\alpha$$

for a scalar coefficient $k(\alpha)$ computed in Section 5.

Proof. Since $\mathcal{I}_{\rm iso}$ is one-dimensional, every linear operator that preserves it acts on it by scalar multiplication. Applying this to the operator L gives the claim. The Laplacian statement is the corresponding special case used in the manuscript. ☐

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Independent Researcher, Kaliningrad, Russia
Email address: me@kurpishev.ru